

A FRACTIONAL EIGENVALUE PROBLEM IN \mathbb{R}^N

GIACOMO BOCERANI

Department of Mathematics and Computer Sciences, University of Perugia
 Via Vanvitelli 1,
 06123 Perugia - Italy

DIMITRI MUGNAI

Department of Mathematics and Computer Sciences, University of Perugia
 Via Vanvitelli 1,
 06123 Perugia - Italy

ABSTRACT. We prove that a linear fractional operator with an asymptotically constant lower order term in the whole space admits eigenvalues.

1. INTRODUCTION

It is well known that the spectrum of the Laplace operator in \mathbb{R}^N is purely continuous, that is $\sigma(-\Delta) = [0, \infty)$. On the other hand, considering operators of the form

$$-\Delta + g(x) \text{ or } -\frac{\Delta}{g(x)},$$

where $g : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies suitable growth conditions, one may hope to apply the standard approach in Hilbert spaces and prove the existence of a principal eigenvalue (see [4] and [6], [9], [12], [13] for examples in a Banach setting) or of a diverging sequence of eigenvalues (see [1], or [17] when $g(x) \rightarrow 0$ at infinity).

In the recent paper [10], the authors consider operators of the form

$$-\Delta u + \beta g(x)u,$$

where $g \approx 1$ at infinity, and study the associated spectrum. Of course, in this case the situation is different, since no compactness argument can be invoked, but they prove that eigenvalues do exist. Inspired by their result, we consider a related situation for the following eigenvalue fractional Laplacian problem in \mathbb{R}^N :

$$(1) \quad (-\Delta)^s u + \beta g(x)u = \lambda u, \quad x \in \mathbb{R}^N.$$

Here $N > 2s$, $s \in (0, 1)$, $\beta > 0$ is a parameter and $(-\Delta)^s$ denotes the fractional Laplacian defined through the Fourier transform in the following way: for any $f \in$

1991 *Mathematics Subject Classification.* Primary: 35R11, 45C05; Secondary: 35A15, 35P15, 49R05.

Key words and phrases. Fractional Laplacian, entire solution, eigenvalue problem, asymptotically linear problem.

The second author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and is supported by the GNAMPA Project *Systems with irregular operators*.

$\mathcal{S}(\mathbb{R}^N)$ with Fourier transform $\mathcal{F}f = \hat{f}$, we define, modulo a positive multiplicative constant depending on N and s ,

$$(2) \quad (-\Delta)^s f = \mathcal{F}^{-1}(|\xi|^{2s} \hat{f}(\xi)),$$

see [19]. When u is sufficiently regular, a pointwise expression of the fractional Laplacian is also available, namely

$$(-\Delta^s)u := -C_{N,s} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy,$$

for some $C_{N,s} > 0$. Here, we will not go into further details about the functional setting of the problem, postponing these aspects to Section 2. We only remark that we look for couples (λ, u) with $\lambda \in \mathbb{R}$ and $u \neq 0$ which satisfy (1).

As in [10], we assume to deal with a function g which is not constant and tends to 1 at infinity. More precisely, denoting by $m(S)$ the measure of a set $S \subset \mathbb{R}^N$, we make the following assumptions on g :

$$(g_1) \quad g \in L^\infty(\mathbb{R}^N), \quad 0 \leq g \leq 1, \quad \lim_{|x| \rightarrow \infty} g(x) = 1 \quad \text{and} \quad m(\{x \in \mathbb{R}^N : g(x) < 1\}) > 0.$$

Although the assumptions on g are rather weak, we show that also in this case the operator

$$L_\beta : H^s(\mathbb{R}^N) \rightarrow (H^s(\mathbb{R}^N))' \quad (\text{cfr. Section 2}), \quad u \mapsto (-\Delta^s)u + \beta g(x)u,$$

admits eigenvalues, which are all below β , see Theorem 2.2 below. We emphasize the fact that, in spite of this difference with the case of $-\Delta$ in bounded domains, we prove the existence of a first eigenfunction which preserves the same nice properties of the first eigenfunction of $-\Delta$ in bounded domains. More precisely, we prove that:

- the first eigenvalue λ_1 is simple,
- the associated eigenfunction is strictly positive in \mathbb{R}^N ,
- any eigenfunction associated to any other eigenvalue is nodal, i.e. sign-changing,
- there exists a different minimax characterization of the second eigenvalue.

Of course, as usual when eigenvalues are found via the Krasnoselskii genus, as we do, we don't know if other eigenvalues can show up.

Completing the previous description, in analogy with the case of $-\Delta$ in \mathbb{R}^N , we observe the following property, which is well-known, and which we prove using an elegant Pohozaev identity.

Proposition 1. $(-\Delta)^s$ has no eigenvalues in \mathbb{R}^N , i.e. if $\mu \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^N)$ are such that

$$(3) \quad \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy = \mu \int_{\mathbb{R}^N} u \varphi dx$$

for all φ in \mathbb{R}^N , then $u \equiv 0$.

Remark 1. Of course, we reduce to problem (3) when $g = 1$ in \mathbb{R}^N , and this shows that the condition " $m(\{x \in \mathbb{R}^N : g(x) < 1\}) > 0$ " in (g_1) is necessary and sufficient to get eigenfunctions for L_β .

The spectral properties that we prove here are the starting point of further investigations about existence results for nonlinear fractional problems in the whole of \mathbb{R}^N , see [3].

2. FUNCTIONAL SETTING AND EIGENVALUES

As usual, we are interested in functions satisfying (1) in a weak sense. For this, let us recall some definitions: for $s \in (0, 1)$ we denote by $H^s(\mathbb{R}^N)$ the Sobolev space of fractional order s defined as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

and

$$[u]_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}$$

denotes the Gagliardo seminorm of u . Moreover, we set $\|\cdot\|_2 = \|\cdot\|_{L^2(\mathbb{R}^N)}$.

Now, by the Plancharel Theorem, $\|u\|_2 = \|\hat{u}\|_2$, and by (2)

$$(4) \quad \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi$$

for every $u \in H^s(\mathbb{R}^N)$. As a consequence, β being positive, $H^s(\mathbb{R}^N)$ can be endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \beta \int_{\mathbb{R}^N} u^2 dx$$

for all $u \in H^s(\mathbb{R}^N)$, see [5].

On the other hand, by [8, Lemma 3.1],

$$(5) \quad \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = c_{N,s} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

for every $u \in H^s(\mathbb{R}^N)$ and some positive constant $c_{N,s}$. Then, by (4) and (5), after setting $c_{N,s} = 1$, we immediately get

$$(6) \quad \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy.$$

In light of the previous considerations, we are now ready to give the following

Definition 2.1. A function $u \in H^s(\mathbb{R}^N)$ is an eigenfunction of problem (1) with associated eigenvalue λ if $u \neq 0$ and

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx + \beta \int_{\mathbb{R}^N} g(x) u \varphi dx = \lambda \int_{\mathbb{R}^N} u \varphi dx$$

for every $\varphi \in H^s(\mathbb{R}^N)$.

Of course, in view of (6), the previous identity can be written as

$$(7) \quad \begin{aligned} & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \\ & + \beta \int_{\mathbb{R}^N} g(x) u \varphi dx = \lambda \int_{\mathbb{R}^N} u \varphi dx \quad \forall \varphi \in H^s(\mathbb{R}^N). \end{aligned}$$

In order to state our main result, let us introduce some notions. First, we set

$$\Sigma = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 dx = 1 \right\}$$

and

$$\lambda_1 = \inf_{u \in \Sigma} \Phi(u),$$

where

$$\Phi(u) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x)u^2 dx.$$

Then, for every $k \in \mathbb{N}$, we introduce the families of sets

$$\Sigma_k = \left\{ A \subset \Sigma_1 : A \text{ is compact, } -A = A \text{ and } \gamma(A) \geq k \right\},$$

where $\gamma(A)$ denotes the Krasnoselskii genus of a symmetric set A , defined as

$$\gamma(A) = \begin{cases} \inf \left\{ m : \exists h \in C^0(A, \mathbb{R}^m) \text{ with } h(u) = -h(-u) \right\}, \\ \infty, \text{ if } \{ \dots \} = \emptyset, \text{ in particular if } 0 \in A, \end{cases}$$

see [16]. Associated to Σ_k we set

$$\lambda_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \Phi(u).$$

Finally, we define

$$M = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 - g(x))|u|^2 dx = 1 \right\},$$

$$M_k = \left\{ A \subset M : A \text{ is compact, } -A = A \text{ and } \gamma(A) \geq k \right\}, \quad k \in \mathbb{N},$$

and

$$\Gamma_k = \inf_{A \in M_k} \sup_{u \in A} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad k \in \mathbb{N}.$$

Note that $M \neq \emptyset$ by (g_1) .

Now, we are ready to state our main result:

Theorem 2.2. *Assume (g_1) . Then*

- (1) $\{\lambda_k\}_k$ is a nondecreasing sequence and $\lambda_k \leq \beta$ for all $k \in \mathbb{N}$.
- (2) If $\lambda_k < \beta$, then λ_k is an eigenvalue of the operator L_β , namely there exist an eigenfunction $u_k \in \Sigma_k$ such that for any $\varphi \in H^s(\mathbb{R}^N)$

$$(8) \quad \begin{aligned} & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_k(x) - u_k(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \\ & + \beta \int_{\mathbb{R}^N} g(x)u_k\varphi dx = \lambda_k \int_{\mathbb{R}^N} u_k\varphi dx. \end{aligned}$$

- (3) If λ_1 is an eigenvalue, then it is simple and the associated eigenfunction e_1 is strictly positive. Moreover, eigenfunctions associated to eigenvalues different from λ_1 are nodal, i.e. they change sign.
- (4) If $\Gamma_k < \beta$, then $\lambda_k < \beta$, and hence λ_k is an eigenvalue.

Proof. (1) Obviously, $\Sigma_1 \supset \Sigma_2 \supset \dots \supset \Sigma_k \supset \dots$, and from the very definition of λ_k we have that $\{\lambda_k\}_k$ is a nondecreasing sequence. Now, let us take $A \in \Sigma_k$, and define

$$A_t = \left\{ v_t : v_t(x) = t^{\frac{N}{2}}u(tx), u \in A, t > 0 \right\}.$$

Then, for all $v \in A_t$, we have

$$\int_{\mathbb{R}^N} |v_t(x)|^2 dx = \int_{\mathbb{R}^N} |u(y)|^2 dy$$

and

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_t(x) - v_t(y)|^2}{|x - y|^{N+2s}} dx dy = t^{2s} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(w) - u(z)|^2}{|w - z|^{N+2s}} dw dz.$$

So, we have

$$\begin{aligned}\Phi(v_t) &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_t(x) - v_t(y)|^2}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x) v_t^2 dx \\ &= t^{2s} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(w) - u(z)|^2}{|w - z|^{N+2s}} dw dz + \beta \int_{\mathbb{R}^N} g\left(\frac{y}{t}\right) u^2 dy.\end{aligned}$$

Since, from Lebesgue's dominated convergence theorem,

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} g\left(\frac{y}{t}\right) u^2 dy = \int_{\mathbb{R}^N} u^2 dy = 1,$$

we immediately have that

$$\lim_{t \rightarrow 0^+} \Phi(v_t) = \beta.$$

Let us notice that such a limit is uniform in $u \in A$, since $A \in \Sigma_k$ is compact. Moreover, also $A_t \in \Sigma_k$ for every $t > 0$, so that $\lambda_k \leq \sup_{v \in A_t} \Phi(v)$, and, as a consequence,

$$\lambda_k \leq \lim_{t \rightarrow 0^+} \sup_{v \in A_t} \Phi(v) = \beta,$$

and the claim holds.

(2) Let us suppose that $\lambda_k < \beta$. If $\Phi|_{\Sigma}$ satisfies the (PS) condition¹, then λ_k is a critical value of $\Phi|_{\Sigma}$ (see [2, Theorem 10.9]), and there exists $u \in \Sigma_k$ such that

$$\begin{aligned}(9) \quad \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x) u \varphi dx \\ = \lambda_k \int_{\mathbb{R}^N} u \varphi dx\end{aligned}$$

for all $\varphi \in H^s(\mathbb{R}^N)$.

Now, suppose that (PS) does not hold. By the Ekeland Principle (see [20, Theorem 8.5]) we can find a (PS) sequence for $\Phi|_{\Sigma_k}$ at level λ_k : this means that there exist a sequence $\{u_n\}_n \subset \Sigma_k$ and a sequence $\{\mu_n\}_n \subset \mathbb{R}$ such that

$$(10) \quad \Phi(u_n) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x) u_n^2 dx \rightarrow \lambda_k$$

as $n \rightarrow \infty$, and

$$\begin{aligned}(11) \quad \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x) u_n \varphi dx \\ - \mu_n \int_{\mathbb{R}^N} u_n \varphi dx = o(1) \|\varphi\|\end{aligned}$$

for all $\varphi \in H^s(\mathbb{R}^N)$.

By (10), it is clear that $\{u_n\}_n$ is bounded in $H^s(\mathbb{R}^N)$. Thus, taking $\varphi = u_n$ in (11), we obtain

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x) u_n^2 dx - \mu_n \int_{\mathbb{R}^N} u_n^2 dx \right) = 0.$$

¹We recall that $\Phi|_{\Sigma_k}$ satisfies the (PS) condition if any sequence $\{u_n\}_n$ in Σ_k such that $\Phi|'_{\Sigma_k}(u_n) \xrightarrow{n \rightarrow \infty} 0$ and $\{\Phi(u_n)\}_n$ is bounded, admits a convergent subsequence.

This limit, together with (10) and the fact that $\{u_n\}_n \subset \Sigma_k$, implies that

$$\lim_{n \rightarrow \infty} \mu_n = \lambda_k.$$

Since $H^s(\mathbb{R}^N)$ is a Hilbert space, we can assume that

$$(12) \quad \begin{aligned} u_n &\rightharpoonup u \in H^s(\mathbb{R}^N) \\ u_n &\rightarrow u \in L_{\text{loc}}^q(\mathbb{R}^N) \text{ for all } q \in [1, 2^{s_*}), \\ u_n &\rightarrow u \text{ a.e in } \mathbb{R}^N, \end{aligned}$$

where $2^{s_*} = \frac{2N}{N-2s}$ is the critical fractional Sobolev exponent (see [15]).

Now, fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$. From (12) we have

$$\begin{aligned} \int_{\mathbb{R}^N} g(x) u_n \varphi \, dx &\rightarrow \int_{\mathbb{R}^N} g(x) u \varphi \, dx, \\ \mu_n \int_{\mathbb{R}^N} u_n \varphi \, dx &\rightarrow \lambda_k \int_{\mathbb{R}^N} u \varphi \, dx \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dxdy \\ = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dxdy. \end{aligned}$$

Thus (11) implies

$$\begin{aligned} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dxdy + \beta \int_{\mathbb{R}^N} g(x) u \varphi \, dx \\ = \lambda_k \int_{\mathbb{R}^N} u \varphi \, dx \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, and, by density, for all $\varphi \in H^s(\mathbb{R}^N)$. This shows that u solves (1) with $\lambda = \lambda_k$. Now, we show that $u \not\equiv 0$. By the Concentration–Compactness Principle, either there exist $R > 0$ and $\nu > 0$ such that (up to subsequences)

$$(13) \quad \lim_{n \rightarrow \infty} \int_{B_R} u_n^2 \, dx = \nu > 0,$$

or

$$(14) \quad \lim_{n \rightarrow \infty} \int_{B_R} u_n^2 \, dx = 0 \text{ for all } R > 0.$$

If (13) holds, then obviously $u \not\equiv 0$. Thus, suppose that (14) holds. By (g₁) we have that for all $\epsilon > 0$ there exist $R_\epsilon > 0$ such that $0 \leq 1 - g(x) < \epsilon$ if $|x| > R_\epsilon$, and from (14)

$$0 \leq \int_{\mathbb{R}^N} (1 - g(x)) u_n^2 \, dx = \int_{B_{R_\epsilon}} (1 - g(x)) u_n^2 \, dx + \int_{\mathbb{R}^N \setminus B_{R_\epsilon}} (1 - g(x)) u_n^2 \, dx.$$

Now,

$$0 \leq \int_{B_{R_\epsilon}} (1 - g(x)) u_n^2 \, dx \leq \int_{B_{R_\epsilon}} u_n^2 \, dx \rightarrow 0$$

as $n \rightarrow \infty$ by assumption, while

$$0 \leq \int_{\mathbb{R}^N \setminus B_{R_\epsilon}} (1 - g(x)) u_n^2 \, dx \leq \epsilon \int_{\mathbb{R}^N \setminus B_{R_\epsilon}} u_n^2 \, dx \leq \epsilon,$$

so that

$$(15) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (1 - g(x)) u_n^2 dx = 0.$$

From (15) we immediately get

$$(16) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} g(x) u_n^2 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (g(x) - 1) u_n^2 dx + 1 = 1.$$

Choosing $\varphi = u_n$ as test function in (11) and passing to the limit, we obtain $\beta = \lambda_k - c$ for some constant $c \geq 0$. Hence, $\beta \leq \lambda_k$, which contradicts our assumption.

Summing up, we have shown that λ_k is an eigenvalue with associated eigenfunction $u \not\equiv 0$.

(3) First, let us notice that we can always find a positive eigenfunction associated to λ_1 . Indeed, for all $u \in H^s(\mathbb{R}^N)$, by the triangle inequality we have

$$\begin{aligned} \Phi(u) &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x) u^2 dx \\ &\geq \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\|u(x) - u(y)\|^2}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x) u^2 dx = \Phi(|u|). \end{aligned}$$

Thus, if e_1 is an eigenfunction associated to λ_1 , and without loss of generality we can assume $|e_1|_p = 1$, we have that

$$\lambda_1 = \Phi(e_1) \geq \Phi(|e_1|) \geq \inf_{|u| \in \Sigma} \Phi(u) \geq \lambda_1,$$

so that also $|e_1| \geq 0$ is an eigenfunction, as well. Thus, $|e_1|$ is a solution of (7), and by the regularity result of [7, Theorem 3.4], we get that $|e_1| \in C^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0, 1)$. Moreover, thanks to the Harnack's inequality in [18, Theorem 3.1], we conclude that $|e_1|$ is strictly positive in \mathbb{R}^N . This implies that all signed solutions of (7) with $k = 1$ are strictly positive (or negative) in the whole of \mathbb{R}^N .

Now let u be another eigenfunction associated to λ_1 . Then, for all $R > 0$ there exists $\chi_R \in \mathbb{R}$ such that

$$\int_{B_R} e_1 dx = \chi_R \int_{B_R} u dx,$$

that is

$$\int_{B_R} (e_1 - \chi_R u) dx = 0.$$

Since $e_1 - \chi_R u$ can be assumed to be a signed eigenfunction associated to λ_1 , then the previous identity implies that $e_1 - \chi_R u = 0$. Now, if $R_1 < R_2$, we find $e_1 = \chi_{R_1} u$ in B_{R_1} and $e_1 = \chi_{R_2} u$ in B_{R_2} . Hence, $\chi_{R_2} = \chi_{R_1} = \chi$. This implies that $e_1 = \chi u$ in the whole space \mathbb{R}^N , i.e. the first eigenvalue is simple.

Finally, let us suppose that ϕ is an eigenfunction associated to an eigenvalue $\lambda > \lambda_1$, and let us suppose that ϕ is signed, for example ϕ is nonnegative, that is $\phi_- := \max\{0, -\phi\} \equiv 0$. We know that if v, w are eigenfunctions associated to eigenvalues $\mu \neq \nu$, then

$$\langle L_\beta v, w \rangle = \mu \int_{\mathbb{R}^N} vw dx \quad \langle L_\beta w, v \rangle = \nu \int_{\mathbb{R}^N} wv dx,$$

so that, using the fact that $\langle L_\beta v, w \rangle = \langle L_\beta w, v \rangle$, we get

$$\int_{\mathbb{R}^N} wv \, dx = 0.$$

In particular, we have

$$\int_{\mathbb{R}^N} e_1 \phi \, dx = 0,$$

which is absurd, because both e_1 and ϕ are nonnegative and non zero. Then $\phi_- \not\equiv 0$. A similar argument can be used to prove that $\phi_+ \not\equiv 0$. In conclusion, ϕ is sign changing.

(4) Let us suppose that $\Gamma_k < \beta$. By definition of Γ_k , there exist a compact and symmetric set $A \in M_k$ such that

$$(17) \quad \sup_{u \in A} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy < \beta.$$

Let us define

$$A^* = \left\{ u^* : u^* = \frac{u}{\|u\|_2}, u \in A \right\}.$$

Of course, $A^* \subset \Sigma$ and $A^* \in \Sigma_k$, in fact A^* is compact and $\gamma(A^*) \geq k$. Now, for any $u^* \in A^*$, we have

$$\begin{aligned} \Phi(u^*) &= \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy + \beta \int_{\mathbb{R}^N} g(x)u^2 \, dx}{\int_{\mathbb{R}^N} u^2 \, dx} \\ &< \frac{\beta \left(1 + \int_{\mathbb{R}^N} g(x)u^2 \, dx \right)}{\int_{\mathbb{R}^N} u^2 \, dx} = \beta, \end{aligned}$$

by (17), recalling that $\int_{\mathbb{R}^N} (1 - g(x))u^2 \, dx = 1$, since $u \in A \in M_k$. But A^* is compact, so

$$\sup_{u^* \in A^*} \Phi(u^*) < \beta,$$

and thus

$$\lambda_k \leq \sup_{u^* \in A^*} \Phi(u^*) < \beta.$$

□

We conclude this section with the

Proof of Proposition 1. Suppose $u \in H^s(\mathbb{R}^N)$ is a solution of (3), then u satisfies the Pohozaev identity

$$\frac{N-2s}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi = N\mu \int_{\mathbb{R}^N} \frac{u^2}{2} \, dx,$$

(see [14, Proposition 4.1]). By (5), we immediately get

$$\frac{N\mu}{2} \int_{\mathbb{R}^N} u^2 \, dx = \mu \frac{N-2s}{2} \int_{\mathbb{R}^N} u^2 \, dx.$$

If $u \not\equiv 0$ and $\mu \neq 0$, then

$$\frac{N\mu}{2} = \mu \frac{N-2s}{2},$$

that is $2s = 0$, which is absurd. If $\mu = 0$, we get $[u]_{H^s(\mathbb{R}^N)} = 0$, so that $\hat{u} \equiv 0$, and so $u \equiv 0$. \square

2.1. Another characterization for λ_2 . For further investigations where the spectrum of L_β plays a rôle, we think it is useful to provide another characterization of the second eigenvalue λ_2 in terms of minimax values:

Proposition 2. *Let us set*

$$\Sigma^* = \left\{ h : h \in C([0, 1], \Sigma) \text{ and } -h(1) = h(0) \geq 0 \right\}$$

and

$$\lambda^* = \inf_{h \in \Sigma^*} \sup_{t \in [0, 1]} \Phi(h(t)).$$

Then $\lambda^* = \lambda_2$.

Proof. Take $h \in \Sigma^*$ and define the map $\tilde{h} : S^1 \rightarrow \Sigma$ by

$$\tilde{h}(e^{i\theta}) = \begin{cases} h\left(\frac{\theta}{\pi}\right) & \text{if } 0 \leq \theta \leq \pi \\ -h\left(\frac{\theta}{\pi} - 1\right) & \text{if } \pi \leq \theta \leq 2\pi. \end{cases}$$

Set $A = \tilde{h}(S^1)$; then $A \in \Sigma_2$. In fact, A is compact, because \tilde{h} is continuous; moreover, A is symmetric, for \tilde{h} is odd; finally, $\gamma(A) \geq 2$: if we suppose by contradiction that $\gamma(A) = 1$, there should exist an odd function $\mathcal{H} \in C^0(A, \mathbb{R} \setminus \{0\})$, but A is a connected set, being the image of a connected set through a continuous function, while $\mathcal{H}(A)$ is disconnected, and a contradiction arises.

So, by definition of λ_2 , we have that

$$(18) \quad \lambda_2 \leq \sup_{u \in A} \Phi(u) = \sup_{t \in [0, 1]} \Phi(h(t)),$$

and hence

$$\lambda_2 \leq \inf_{h \in \Sigma^*} \sup_{t \in [0, 1]} \Phi(h(t)) = \lambda^*.$$

By Theorem 2.2.(1), we know that $\lambda_2 \leq \beta$, and using arguments similar to those used therein, we can prove that $\lambda^* \leq \beta$. Thus, if $\lambda_2 = \beta$, then $\lambda_2 = \lambda^* = \beta$.

On the other hand, if $\lambda_2 < \beta$, by Theorem 2.2.(2), (3), there exist a sign-changing eigenfunction u , corresponding to the eigenvalue λ_2 . Let $u = u_+ - u_-$, with $u_+, u_- \not\equiv 0$, and where $u_+ := \max\{u, 0\}$. Taking $\varphi = u_\pm$ as test function in (8) with $k = 2$, we have

$$(19) \quad \begin{aligned} & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(u_\pm(x) - u_\pm(y))}{|x - y|^{N+2s}} dx dy \\ & + \beta \int_{\mathbb{R}^N} g(x) u u_\pm dx = \lambda_2 \int_{\mathbb{R}^N} u u_\pm dx. \end{aligned}$$

Writing $u = u_+ - u_-$, we can observe that

$$\begin{aligned} & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(u_+(x) - u_+(y))}{|x - y|^{N+2s}} dx dy \\ &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_+(x) - u_+(y)|^2}{|x - y|^{N+2s}} dx dy + 2 \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u_+(x)u_-(y)}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Using a similar argument with u_- , from (19) we have that

$$\begin{aligned} (20) \quad \Phi(u_{\pm}) &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_{\pm}(x) - u_{\pm}(y)|^2}{|x - y|^{N+2s}} dx dy + \beta \int_{\mathbb{R}^N} g(x)u_{\pm}^2 dx \\ &= -2 \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u_+(x)u_-(y)}{|x - y|^{N+2s}} dx dy + \lambda_2 \int_{\mathbb{R}^N} u_{\pm}^2 dx. \end{aligned}$$

Now, set $h : [0, 1] \rightarrow \Sigma$ by

$$h(t) = \frac{u_+ \cos(\pi t) + u_- \sin(\pi t)}{\|u_+ \cos(\pi t) + u_- \sin(\pi t)\|_2}.$$

Obviously, h is continuous and

$$h(0) = \frac{u_+}{\|u_+\|_2}, \quad h(1) = -\frac{u_+}{\|u_+\|_2},$$

so $h \in \Sigma^*$. Let us compute $\Phi(h(t))$:

$$\begin{aligned} \Phi(h(t)) &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{((u_+(x) \cos(\pi t) + u_-(x) \sin(\pi t)) - ((u_+(y) \cos(\pi t) + u_-(y) \sin(\pi t))^2)}{\|u_+ \cos(\pi t) + u_- \sin(\pi t)\|_2^2 |x - y|^{N+2s}} dx dy \\ &\quad + \beta \frac{\cos^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N} g(x)|u_+|^2 dx + \beta \frac{\sin^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N} g(x)|u_-|^2 dx. \\ &= \frac{\cos^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_+(x) - u_+(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{\sin^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_-(x) - u_-(y)|^2}{|x - y|^{N+2s}} dx dy \\ &\quad - \frac{4 \cos(\pi t) \sin(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u_+(x)u_-(y)}{|x - y|^{N+2s}} dx dy + \beta \frac{\cos^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N} g(x)|u_+|^2 dx \\ &\quad + \beta \frac{\sin^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N} g(x)|u_-|^2 dx. \end{aligned}$$

By (20), we get

$$\begin{aligned} (21) \quad \Phi(h(t)) &= \frac{\cos^2(\pi t)}{\|...\|_2^2} \left(\lambda_2 \int_{\mathbb{R}^N} u_+^2 dx - 2 \int_{\mathbb{R}^N} \frac{u_+(x)u_-(y)}{|x - y|^{N+2s}} dx \right) \\ &\quad + \frac{\sin^2(\pi t)}{\|...\|_2^2} \left(\lambda_2 \int_{\mathbb{R}^N} u_-^2 dx - 2 \int_{\mathbb{R}^N} \frac{u_+(x)u_-(y)}{|x - y|^{N+2s}} dx \right) \\ &\quad - 4 \frac{\cos(\pi t) \sin(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N} \frac{u_+(x)u_-(y)}{|x - y|^{N+2s}} dx. \end{aligned}$$

But

$$(22) \quad u_+(x)u_-(y) (\cos^2(\pi t) + \sin^2(\pi t) + 2 \cos(\pi t) \sin(\pi t)) \geq 0,$$

so that (21) and (22) imply that

$$\Phi(h(t)) \leq \lambda_2 \frac{\cos^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N} u_+^2 dx + \lambda_2 \frac{\sin^2(\pi t)}{\|...\|_2^2} \int_{\mathbb{R}^N} u_-^2 dx = \lambda_2.$$

Thus $\Phi(h(t)) \leq \lambda_2$ for all $t \in [0, 1]$, and then

$$\lambda^* \leq \sup_{t \in [0, 1]} \Phi(h(t)) \leq \lambda_2,$$

and hence $\lambda^* = \lambda_2$. □

REFERENCES

- [1] (MR1098396) W. Allegretto, Principal eigenvalues for indefinite-weight elliptic problems in \mathbb{R}^n , *Proc. Amer. Math. Soc.*, **116** (1992), 701–706.
- [2] (MR2292344) A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge, 2007.
- [3] G. Bocerani and D. Mugnai, An asymptotically linear fractional problem in \mathbb{R}^N , submitted.
- [4] (MR1007489) K.J. Brown, C. Cosner and J. Fleckinger, Principal eigenvalues for problems with indefinite weight function on \mathbb{R}^N , *Proc. Amer. Math. Soc.*, **109** (1990), 147–155.
- [5] (MR3007900) X. Chang and Z.-Q. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, *Nonlinearity*, **26** (2013), 479–494.
- [6] (MR1390979) P. Drábek and Y. X. Huang, Bifurcation problems for the p -Laplacian in \mathbb{R}^n , *Trans. Amer. Math. Soc.*, **349**, (1997), 171–188.
- [7] (MR3002595) P. Felmer, A. Quaas and J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A*, **142** (2012), 1237–1262.
- [8] (MR2425175) R. L. Frank, E. H. Lieb and R. Seiringer, Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators, *J. Amer. Math. Soc.*, **21** (2008), 925–950.
- [9] (MR2677824) L. Leadi and A. Yechoui, Principal eigenvalue in an unbounded domain with indefinite potential, *NoDEA Nonlinear Differential Equations Appl.*, **17** (2010), 391–409.
- [10] (MR2733241) J. Liu, X. Liu and Y. Guo, On an asymptotically p -linear p -Laplacian equation in \mathbb{R}^N , *Nonlinear Anal.*, **74** (2011), 676–688.
- [11] (MR0778974) P. L. Lions, The concentration compactness principle in the calculus of variations, the locally compact case, II, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 223–283.
- [12] (MR3283555) D. Mugnai and N. S. Papageorgiou, Bifurcation for positive solutions of nonlinear diffusive logistic equations in \mathbb{R}^N with Indefinite Weight, *Indiana Univ. Math. J.*, **63** (2014), 1397–1418.
- [13] (MR3060699) D. Mugnai and N. S. Papageorgiou, Resonant nonlinear Neumann problems with indefinite weight, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, **XI** (2012), 729–788.
- [14] S. Secchi, On Fractional Schrödinger Equations In \mathbb{R}^N without the Ambrosetti-Rabinowitz Condition, *Topol. Methods. Nonlinear Anal.*, to appear, arXiv 1210.0755.
- [15] (MR0290095) E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [16] (MR2431434) M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, Berlin, 2008.
- [17] (MR1690753) A. Szulkin and M. Willem, Eigenvalue problems with indefinite weight, *Studia Mathematica*, **135** (1999), 191–201.
- [18] (MR2825646) J. Tan and J. Xiong, A Harnack inequality for fractional Laplace equations with lower order terms, *Disc. Cont. Dyn. Syst.*, **31** (2011) 975–983.
- [19] (MR2584076) E. Valdinoci, From the long jump random walk to the fractional Laplacian, *Bol. Soc. Esp. Mat. Apl.*, **49** (2009), 33–44.
- [20] (MR1400007) M. Willem, *Minimax Theorems*, Progr. Nonlinear Differential Equations Appl., vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: giacomo.bocerani@unipg.it

E-mail address: dimitri.mugnai@unipg.it